STATIC, STAGNATION, AND DYNAMIC PRESSURES

Bernoulli equation is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

In this equation **p** is called static pressure, because it is the pressure that would be measured by an instrument that is static with respect to the fluid. Of course, if the instrument were static with respect to flowing fluid, it would have move along with the fluid. However, such a measurement rather difficult to make in a practical situation. **However, we showed that there was no pressure variation normal to straight streamlines.** This fact makes it possible to measure the static pressure in a flowing fluid using a **wall pressure "tap"** placed in a region where the flow streamlines are straight as shown in the figure. The pressure tap is a small hole, drilled carefully in the wall, with its axis perpendicular to the surface.

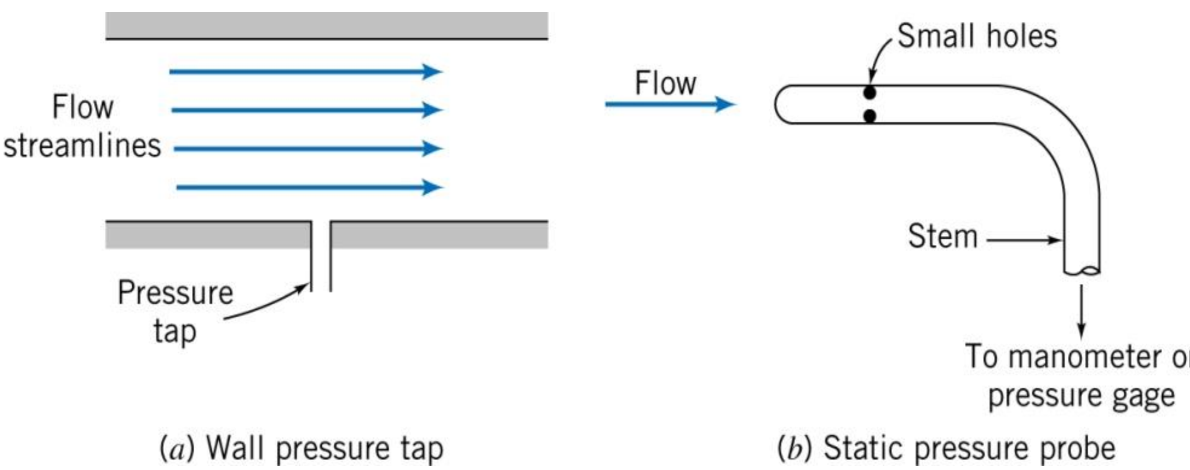


Figure. Measurement of static pressure.

In a fluid stream far from a wall, or where streamlines are curved, accurate static pressure measurements can be made **by careful use of a static pressure probe**, shown in the figure.

When a flowing fluid is decelerated to zero speed by a frictionless process, the pressure measured at that point is called **stagnation pressure**.

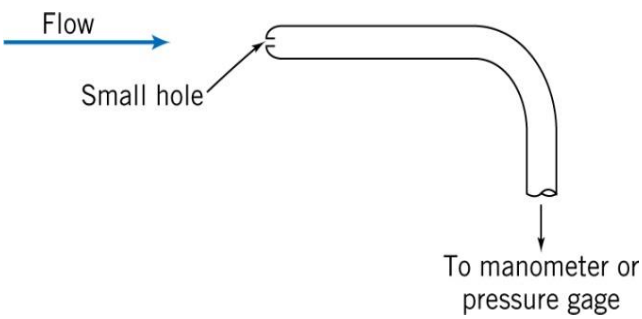


Figure. Measurement of stagnation pressure (Pitot tube).

In incompressible flow, applying Bernoulli equation between points in the free stream and at the nose of tube and taking *z = 0* at the tube centerline, we get

$$\frac{p_0}{\rho} + \underbrace{\frac{V_0^2}{2}}_{=0} = \frac{p}{\rho} + \frac{V^2}{2}$$

where P_0 is the stagnation pressure, the stagnation speed V_0 is zero.

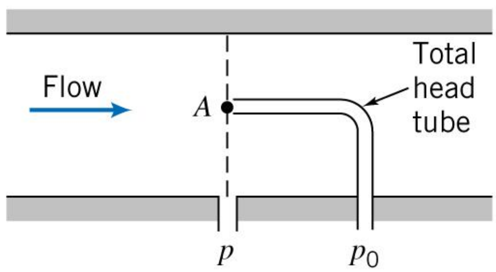
$$\therefore \qquad p_0 = p + \frac{1}{2}\rho V^2$$

where p is the static pressure. The term $\frac{1}{2}\rho V^2$ generally is called **dynamic pressure**. Solving the dynamic pressure, we get,

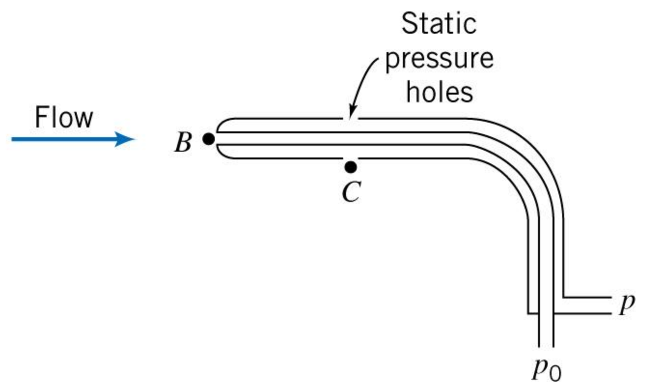
$$\frac{1}{2}\rho V^2 = p_0 - p$$

and for the speed

$$V = \sqrt{\frac{2(p_0 - p)}{\rho}}$$



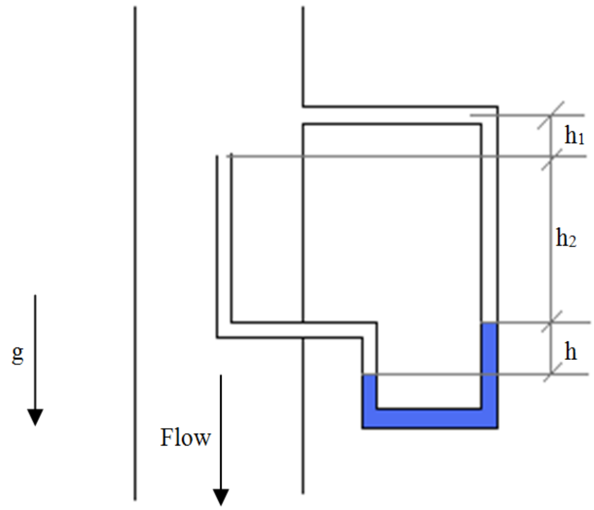
(a) Total head tube used with wall static tap



(b) Pitot-static tube

The **static pressure** corresponds to a point **A** is read from the wall static pressure tap. The **stagnation pressure** is measured directly at **A** by the **total head tube**.	Two probes are combined as in **pitot-static tube**. The inner tube is used to measure the **stagnation pressure** at point **B** while the **static pressure at C** is measured by the small holes in the outer tube.

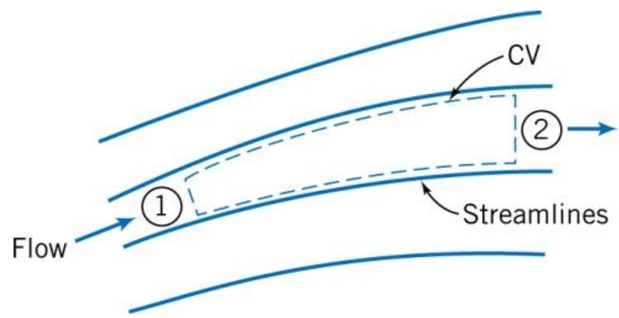
Example: A simple pitot tube and a piezometer are installed in a vertical pipe as shown in the figure. If the deflection in the mercury manometer is **0.1 m**, then **determine the velocity of the water at the center of the pipe.** The densities of water and mercury are **1000 kg/m³** and **13600 kg/m³**, respectively.



To be completed in class

RELATION BETWEEN THE FIRST LAW OF THERMODYNAMICS AND THE BERNOULLI EQUATION

Consider steady flow in the absence of shear forces. We choose a control volume **bounded by streamlines along its periphery**. Such a control volume often is called a **streamtube**. We apply energy equation to this control volume.



Basic equation (Energy equation)

$$\dot{Q} - \underbrace{\dot{W}_s}_{0} - \underbrace{\dot{W}_{shear}}_{0} - \underbrace{\dot{W}_{other}}_{0} = \underbrace{\frac{\partial}{\partial t}\int_{C\forall} e\rho d\forall}_{0} + \int_{CS} (e + p\vartheta)\rho\vec{V}\cdot d\vec{A}$$

Restrictions:

1) $\dot{W}_s = 0$
2) $\dot{W}_{shear} = 0$
3) $\dot{W}_{other} = 0$
4) Steady flow
5) Uniform flow and properties at each section

Under these restrictions

$$0 = \left(u_1 + p_1 v_1 + \frac{V_1^2}{2} + gz_1\right)\{-|\rho_1 V_1 A_1|\} + \left(u_2 + p_2 v_2 + \frac{V_2^2}{2} + gz_2\right)\{+|\rho_2 V_2 A_2|\} - \dot{Q}$$

But from continuity under these restrictions

$$0 = \underbrace{\frac{\partial}{\partial t}\int_{C\forall} \rho d\forall}_{0} + \int_{CS} \rho\vec{V}\cdot d\vec{A} \quad \text{or} \quad 0 = \{-|\rho_1 V_1 A_1|\} + \{+|\rho_2 V_2 A_2|\}$$

That is, $\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2$

Also, $\dot{Q} = \frac{\delta Q}{\delta t} = \frac{\delta Q}{dm}\frac{dm}{dt} = \frac{\delta Q}{dm}\dot{m}$

Thus, from the energy equation

$$0 = \left[\left(p_2 v_2 + \frac{V_2^2}{2} + g z_2\right) - \left(p_1 v_1 + \frac{V_1^2}{2} + g z_1\right)\right]\dot{m} + \left(u_2 - u_1 - \frac{\delta Q}{dm}\right)\dot{m}$$

or $$p_1 v_1 + \frac{V_1^2}{2} + g z_1 = p_2 v_2 + \frac{V_2^2}{2} + g z_2 + \left(u_2 - u_1 - \frac{\delta Q}{dm}\right)$$

Under the restriction of **incompressible flow** $v_1 = v_2 = \frac{1}{\rho}$ and hence

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 + \left(u_2 - u_1 - \frac{\delta Q}{dm}\right)$$

This will reduce to the Bernoulli equation if the term in parentheses were zero. Thus, under the additional restrictions,

6) incompressible flow $v_1 = v_2 = \frac{1}{\rho} = \text{constant}$

7) $\left(u_2 - u_1 - \frac{\delta Q}{dm}\right) = 0$

The energy equation reduces to

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 = \text{constant}$$

Before, the Bernoulli equation was derived **from momentum considerations (Newton's second law),** and is valid for steady, incompressible, frictionless flow along a streamline.

In this section, the Bernoulli equation was obtained by applying the **first law of thermodynamics** to a streamtube control volume, subject to restrictions 1 through 7 above.

Example: Consider the frictionless, incompressible flow with heat transfer. Show that

$$u_2 - u_1 = \frac{\delta Q}{dm}$$

To be completed in class

ENERGY GRADE LINE AND HYDRAULIC GRADE LINE

Often it is convenient to represent the mechanical energy level of a flow graphically. **The energy equation, that is Bernoulli equation**, suggests such a representation. Dividing Bernoulli equation by *g*, we obtain

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = H = \text{constant}$$

Each term has dimensions of length, or "**head**" of flowing fluid. The individual terms are

$\frac{p}{\rho g}$ is **the head due to local static pressure**

$\frac{V^2}{2g}$ is **the head due to local dynamic pressure**

z is **elevation head**

H is **the total head of the flow**

The energy grade line (EGL): The locus of points at a vertical distance,

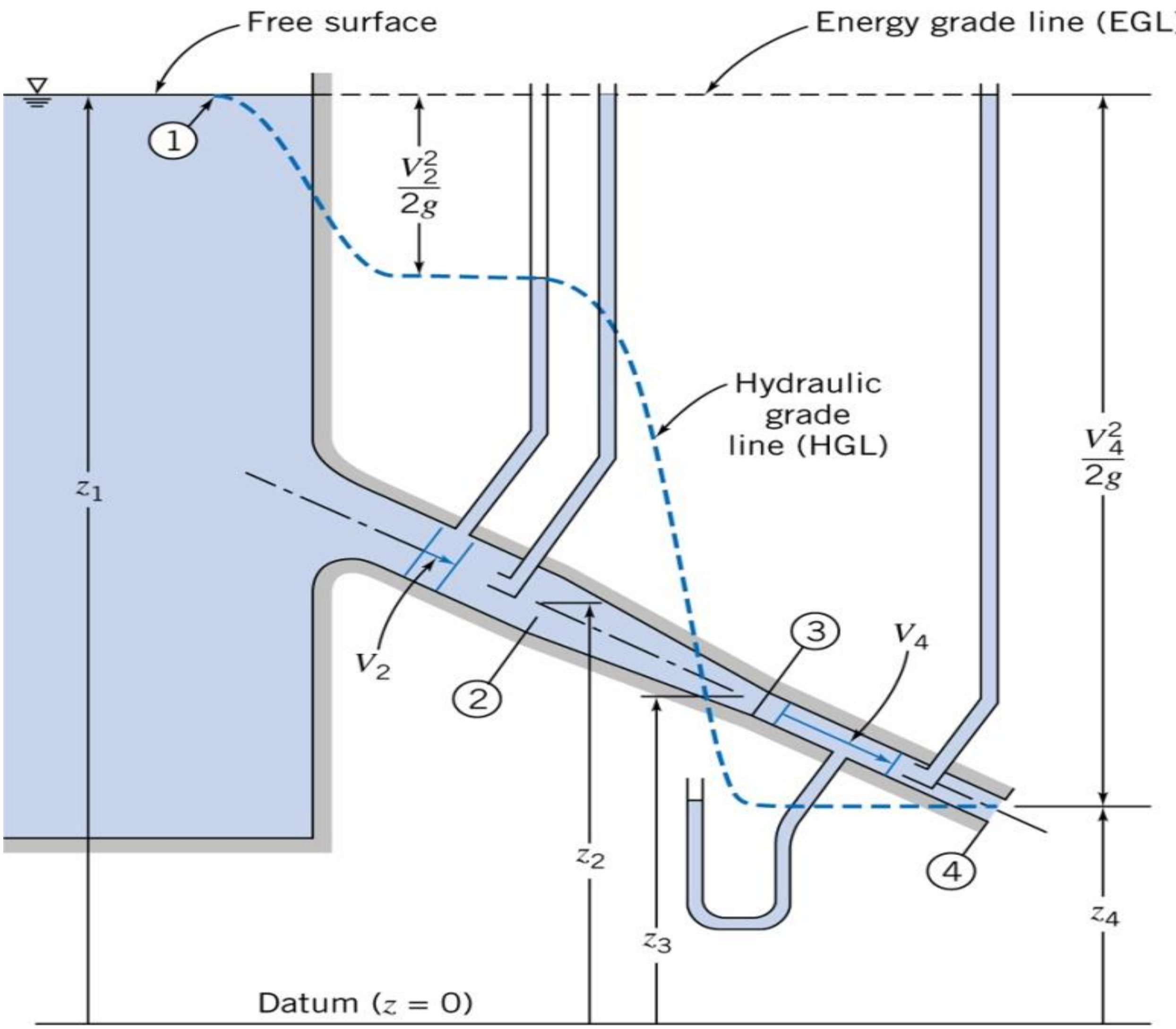
$H = \frac{p}{\rho g} + \frac{V^2}{2g} + z$, measured above a horizontal datum,

which is the total head of the fluid.

The hydraulic grade line (HGL): The locus of points at a vertical distance,

$\frac{p}{\rho g} + z$, measured above a horizontal datum.

The difference is heights between the EGL and HGL represents, **the dynamic (velocity) head,** $\frac{V^2}{2g}$.


Free surface
Energy grade line (EGL)
1
$\frac{V_2^2}{2g}$
Hydraulic
grade
line (HGL)
$\frac{V_4^2}{2g}$
z_1
V_2
2
3
V_4
4
z_2
z_3
z_4
Datum ($z = 0$)

UNSTEADY BERNOULLI EQUATION
– INTEGRATION OF EULER'S EQUATION ALONG A STREAMLINE

Consider the streamwise Euler equation in streamline coordinates

$$V\frac{\partial V}{\partial s}+\frac{1}{\rho}\frac{\partial p}{\partial s}+g\frac{\partial z}{\partial s}+\frac{\partial V}{\partial t}=0$$

The above equation may now be integrated along an instantaneous streamline from point 1 to point 2 to yield

$$\int_1^2 V\frac{\partial V}{\partial s}ds+\int_1^2\frac{1}{\rho}\frac{\partial p}{\partial s}ds+\int_1^2 g\frac{\partial z}{\partial s}ds+\int_1^2\frac{\partial V}{\partial t}ds=0$$

For an incompressible flow, it becomes

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2+\int_1^2\frac{\partial V_s}{\partial t}ds$$

Restrictions:

1) Incompressible flow
2) Frictionless flow
3) Flow along a streamline

Example: A long pipe is connected to a **large reservoir** that initially is filled with water to a depth of **3 m**. The pipe is **150 mm** in diameter and **6 m** long. As a first approximation, friction may be neglected. **Determine the flow velocity leaving the pipe as a function of time after a cap is removed from its free end.** The reservoir is large enough so that the change in its level may be neglected.

To be completed in class

FLOW MEASUREMENT

Flow measurement refers to the ability to measure the velocity, volume flow rate, or mass flow rate of any liquid or gas.

There are many types of devices used for flow measurement. Many of these devices use the rinciple of Bernoulli equation.

The choice of a flow meter is influenced by accuracy required, range, cost, complication, ease of reading or data reduction, and service life.

Flow Measurement Techniques

In general devices used for flow measurement can be grouped depending on the nature of the data obtained by the device. Based on this, flow measurement devices can be grouped as follows:

A) Measurement of Integral Properties of Flows (Mass and volume flow measurement)

1) Restriction flow meters for Internal Flows
    a) Orifice meter
    b) Flow nozzle
    c) Venturi meter
2) Rotameter
3) Turbine flow meter
4) Coriolis technique

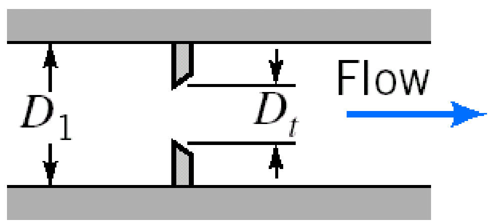
B**) Measurement of Local Flow Parameters (Local Velocity Measurement)**

1) Pitot-static tube
2) Hot wire anemometer
3) Lase doppler anemometry (LDA)
4) Particle image velocimery (PIV)
5) Ultrosonic technique
6) Magnetic technique

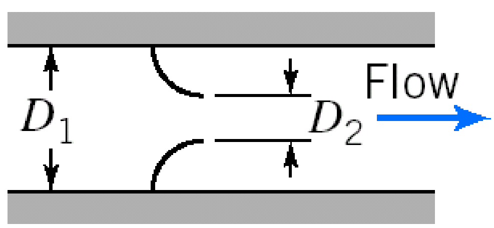
MASS AND VOLUME FLOW MEASUREMENT

Restriction Flow Meters

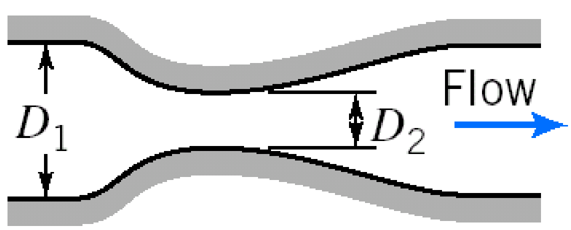
An easy and cheap way to measure flow rate through a pipe is to place some type of restriction within the pipe as shown in the figure below:



Orifice meter
- Head loss high
- Initial cost low



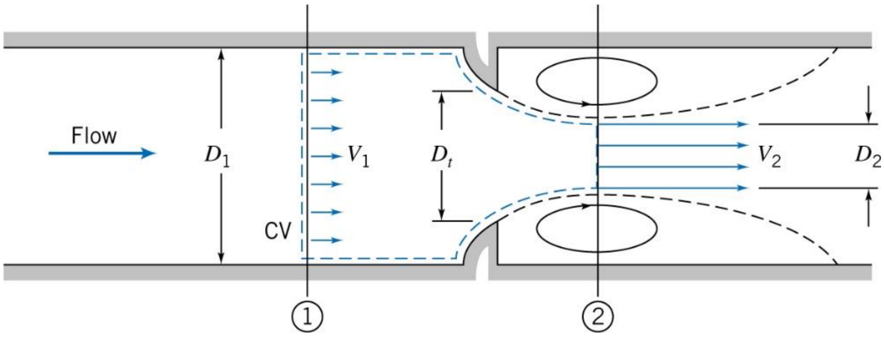
Flow nozzle meter
- Head loss intermediate
- Initial cost intermediate



Venturi meter
- Head loss low
- Initial cost high

The operation of each of these devices is based on the same principles, i.e. due to restriction velocity increases and pressure decreases.

We assume the flow is horizontal (z1=z2), steady, frictionless and incompressible between points **1** and **2**:



Due to sharp edge of flow nozzle and orifice, flow separation and hence recirculating zones forms as seen in the figure. The main stream flow continues to accelerate from nozzle throat and a «**vena contracta**» forms at cross section 2. After cross-section 2, the flow decelerates again and fill the duct.

At **vena contracta**, the flow area is minimum, streamlines are straight, and the pressure is uniform across the channel section.

Theoretical flow rate can be obtained using Bernoulli equation and equation of conservation of mass as follows:

To be completed in class

$$\dot{m}_{theoretical} = \frac{A_2}{\sqrt{1-\left(\frac{A_2}{A_1}\right)^2}}\sqrt{2\rho\left(P_1 - P_2\right)} \qquad (A)$$

This equation shows that under our set of assumptions, for a given fluid (ρ) and flow meter geometry (A1 and A2), the flow rate is directly proportional to the pressure drop across the meter tabs, i.e.

$$\dot{m}_{theoretical} \propto \sqrt{\Delta P}$$

Flow rate at cross-sectional area **2** is unknown when **vena contracta** is pronounced. Frictional effects can become important (especially down stream from the meter.) when the meter contours are abrupt. Finally, the location of the pressure taps influences the differential pressure reading.

Due to the above reasons, the actual flow rate is different from the theoretical flow rate given by Eq. (A). Hence, Eq. (A) is adjusted for Reynolds number and diameter ratio (Dt/D1) by defining and empirical **discharge coefficient C**, as follows:

$$\dot{m}_{actual} = C\dot{m}_{theoretical} = \frac{CA_t}{\sqrt{1-\left(\frac{A_t}{A_1}\right)^2}}\sqrt{2\rho\left(P_1 - P_2\right)}$$

$$Letting \quad \frac{D_t}{D_1} = \beta, \quad then \quad \left(\frac{A_t}{A_1}\right)^2 = \left(\frac{D_t}{D_1}\right)^4 = \beta^4$$

Therefore,

$$\dot{m}_{actual} = \frac{CA_t}{\sqrt{1-\beta^4}}\sqrt{2\rho\left(P_1 - P_2\right)}$$

$$\frac{1}{\sqrt{1-\beta^4}} \quad \text{is known as "velocity of approach factor"}$$

Discharge coefficient **C** and velocity of approach factor are combined into a single «**flow coefficient**» **K** as

$$K = \frac{C}{\sqrt{1-\beta^4}}$$

In terms of flow coefficient, actual mass flow arte is expressed as,

$$\dot{m}_{actual} = KA_t\sqrt{2\rho\left(P_1 - P_2\right)}$$

For standardized meters, test data is used to develop empirical equations that predict the discharge and flow coefficients as a function of diameter ratio Dt/D1 and Reynolds number.

For the turbulent flow regime (Re>4000), discharge coeffient and flow coefficient may me expressed as follows:

$$C = C_\infty + \frac{b}{\mathrm{Re}_{D_1}^n} \qquad\qquad K = K_\infty + \frac{1}{\sqrt{1-\beta^4}}\frac{b}{\mathrm{Re}_{D_1}^n}$$

In the above equations, subscript ∞ denotes the coefficient at infinite Reynolds number; constants **b** and **n** allow for scaling to finite Reynolds number.

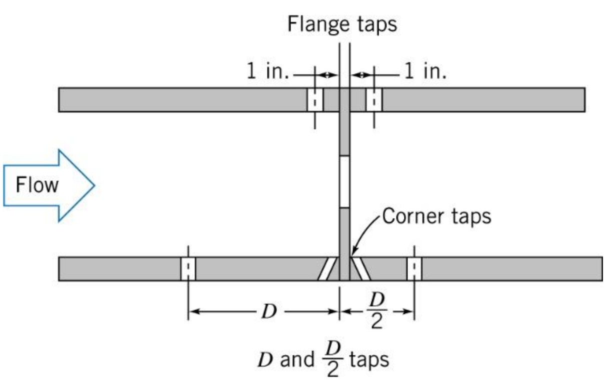
Correlating equations and curves of coefficients versus reynolds number are given for orifice plate, flow nozzle and venturi meter.

Orifice Meter (Plate)

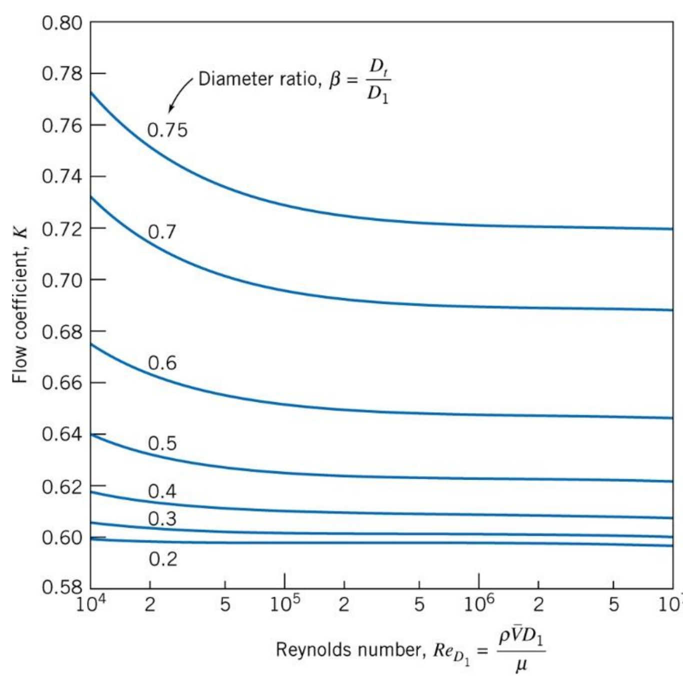
The correlating equation recommended for a concentric orifice with corner tabs is

$$C = 0.5959 + 0.0312\beta^{2.1} - 0.184\beta^{8} + \frac{91.71\beta^{2.5}}{\mathrm{Re}_{D_1}^{0.75}}$$

This equation predicts the discharge coefficient **C** within **±0.6 percent** for **0.2<β <0.75 and for 104 <Re <107**. Flow coefficients calculated from the above equation are presented in figure below:



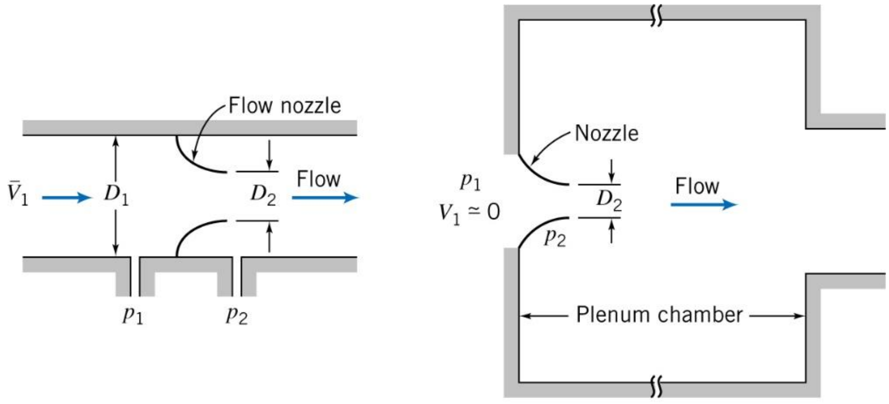
Pressure tabs for orifices may be placed in several locations as shown in above figure.



Flow coefficient for corner concentric orifices with corner taps.

Flow Nozzle

Flow nozzle may be used as metering elements in either plenums or ducts as shown in figure.

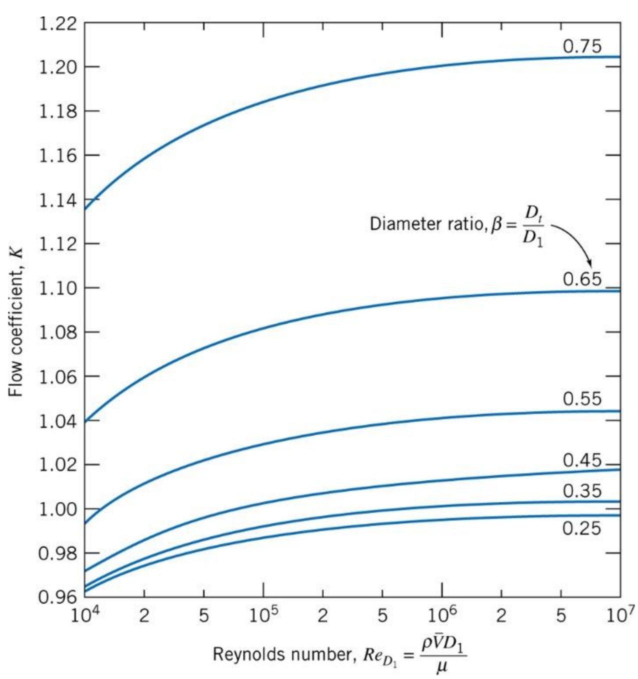


(a) In duct (b) In plenum

Correlating equation recommended for an ASME long-radius flow nozzle is

$$C = 0.9957 - \frac{6.53\beta^{0.5}}{\mathrm{Re}_{D_1}^{0.5}}$$

This equation predicts the discharge coefficient **C** for the flow nozzle within **±2.0 percent** for **0.25<β <0.75 and for 104 <Re <107**. Flow coefficients calculated from the above equation are presented in figure below:



Flow coefficients for ASME long-radius flow nozzle.

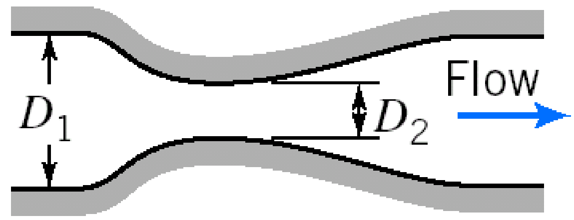
For plenum nozle

β=0.0

Flow coefficient K is in the range of **0.95<K <0.99**

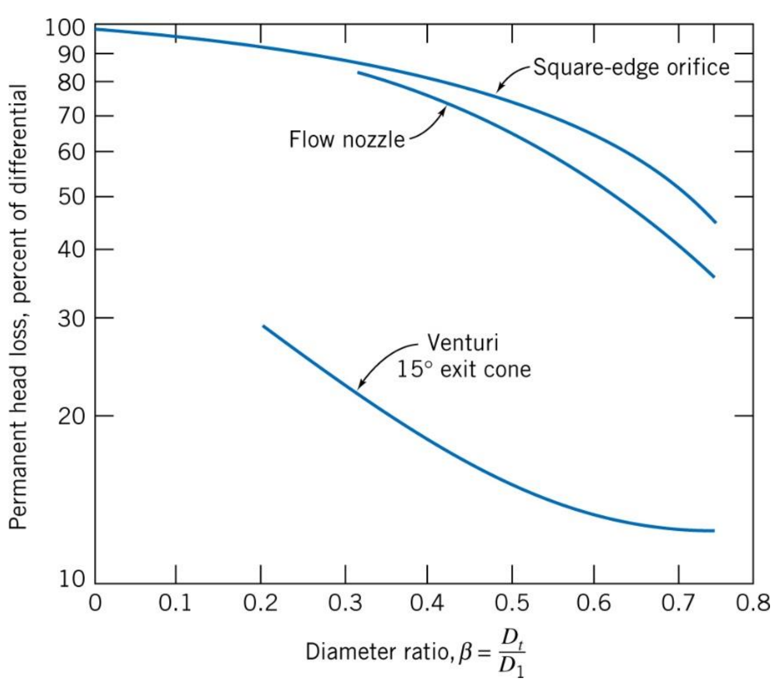
Venturi Meter

Experimental data show that discharge coefficients for venturi meters range from **0.980 to 0.995** at high Reynolds numbers **(ReD1>2x105)** Thus, **C=0.99** can be used to measure the mass flow rate within about **±1 percent** at high Reynolds numbers.

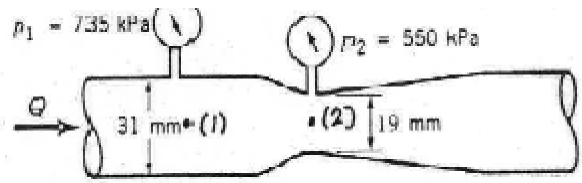


Permanent Head Loss Produced by Flow metering Elements

The unrecoverable loss in head across a metering element may be expressed as a function of the differential pressure Δp, across the element. Pressure losses are displayed as functions of diameter ratio in the figure below:

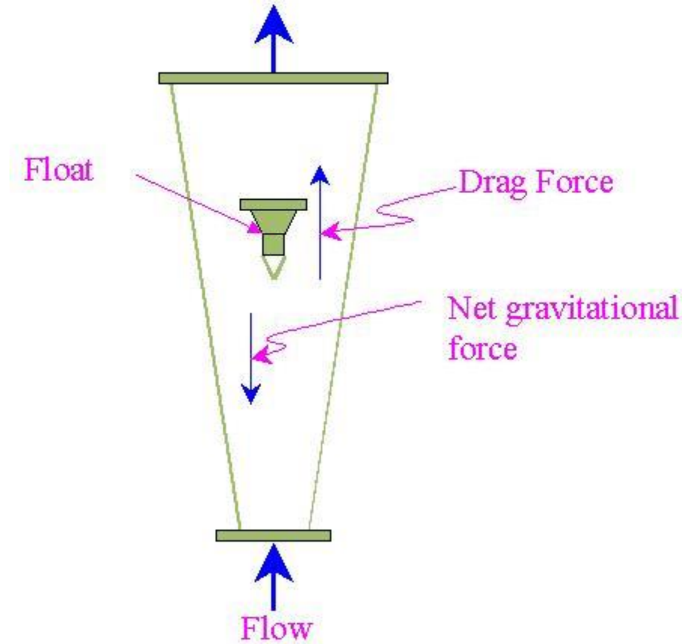


Example: Determine the flow rate of water through the venturi meter shown in the figure. Calculate theoretical and actual flow rates.



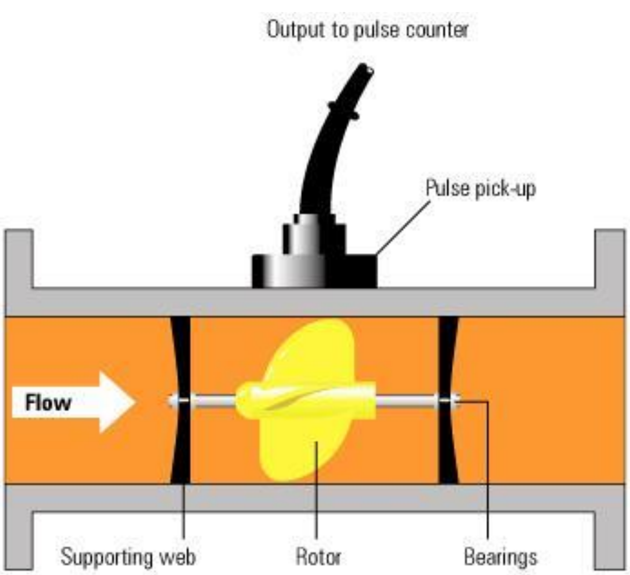
To be completed in class

Rotameter



1. A free moving float is balanced inside a vertical tapered tube
2. As the fluid flows upward the float remains steady when the dynamic forces acting on it are zero.
3. The flow rate indicated by the position of the float relative to a calibrated scale.

Turbine Flow Meter

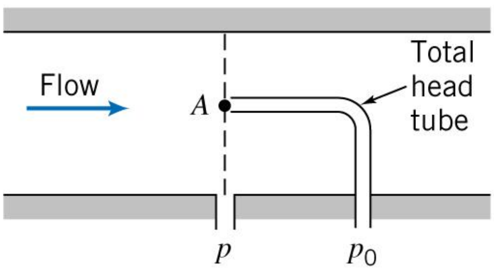


1. Consists of a multi-bladed rotor mounted at right angles to the flow & suspended in the fluid stream on a free-running bearing.
2. The diameter of the rotor is slightly less than the inside diameter of the flow metering chamber.
3. Speed of rotation of rotor proportional to the volumetric flow rate.

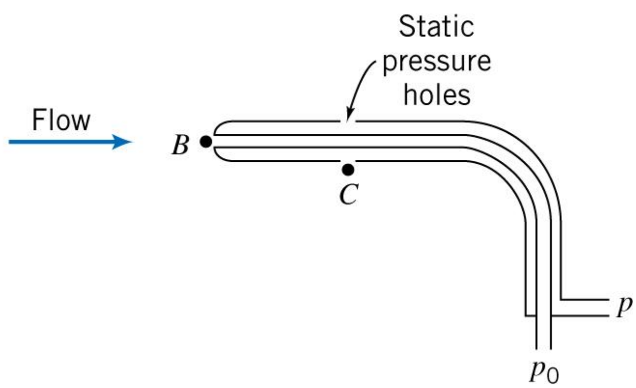
LOCAL VELOCITY MEASUREMENT

1) Pitot-static tube
2) Hot wire anemometer
3) Lase doppler anemometry (LDA)
4) Particle image velocimery (PIV)
5) Ultrosonic technique
6) Magnetic technique

Pitot-Static Tube



(*a*) Total head tube used with wall static tap



(*b*) Pitot-static tube

The **static pressure** corresponds to a point **A** is read from the wall static pressure tap. The **stagnation pressure** is measured directly at **A** by the **total head tube**.	Two probes are combined as in **pitot-static tube**. The inner tube is used to measure the **stagnation pressure** at point **B** while the **static pressure at C** is measured by the small holes in the outer tube.

See pressure measurement technique above

Example: A pitot-static tube is used to measure the speed of air at standard conditions at a point in a flow. The manometer deflection in millimeters of water is measured as **63 mm**. Determine the speed of air at that point.

IRROTATIONAL FLOW

When the fluid elements moving in a flow field do not undergo any rotation, then the flow is known to be **irrotational**. For an irrotational flow,

$\vec{w} = 0$ or $\nabla \times \vec{V} = 0$ that is,

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

In cylindrical coordinates,

$$\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} = \frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta} = 0$$

BERNOULLI EQUATION APPLIED TO IRROTATIONAL FLOW

Euler equation for steady flow was

$$-\frac{1}{\rho}\nabla p - g\nabla z = \left(\vec{V} \cdot \nabla\right)\vec{V}$$

using vector identity

$$\left(\vec{V} \cdot \nabla\right)\vec{V} = \frac{1}{2}\nabla\left(\vec{V} \cdot \vec{V}\right) - \vec{V} \times \left(\nabla \times \vec{V}\right)$$

We see that for irrotational flow $\nabla \times \vec{V} = 0$; therefore, it reduces to

$$\left(\vec{V} \cdot \nabla\right)\vec{V} = \frac{1}{2}\nabla\left(\vec{V} \cdot \vec{V}\right)$$

And **Euler's equation for irrotational** flow can be written as

$$-\frac{1}{\rho}\nabla p - g\nabla z = \frac{1}{2}\nabla\left(\vec{V} \cdot \vec{V}\right) = \frac{1}{2}\nabla\left(V^2\right)$$

During the interval ***dt***, a fluid particle moves from the vector position $\vec{r}$ to the position $\vec{r}+d\vec{r}$. Taking the dot product of $d\vec{r}=dx\vec{i}+dy\vec{j}+dz\vec{k}$ with each of the terms in above equation, we obtain

$$-\frac{1}{\rho}\nabla p\cdot d\vec{r}-g\nabla z\cdot d\vec{r}=\frac{1}{2}\nabla\left(V^2\right)\cdot d\vec{r}$$

and hence

$$-\frac{dp}{\rho}-gdz=\frac{1}{2}d\left(V^2\right)$$

integrating this equation gives,

$$\int\frac{dp}{\rho}+\frac{V^2}{2}+gz=\text{constant}$$

For incompressible flow, ρ = constant, and

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=\text{constant}$$

Since **dr** was an arbitrary displacement, **this equation is valid between any two points in the flow field. The restrictions are**

1. Steady flow
2. Incompressible flow
3. Inviscid flow
4. **Irrotational flow**

VELOCITY POTENTIAL

We can formulate a relation called the **potential function, ϕ**, for a velocity field that is **irrotational**. To do so, we must use the fundamental vector identity

$$\text{curl}\left(\text{grad } \phi\right) = \nabla \times \left(\nabla \phi\right) = 0$$

which is **valid if *ϕ(x,y,z,t)* is a scalar function**, having continuous first and second derivatives.

Then, for an irrotational flow in which $\nabla \times \vec{V} = 0$, a scalar function, ϕ, must exist such that the gradient of ϕ is equal to the velocity vector, $\vec{V}$.

$$\vec{V} \equiv \pm \nabla \phi$$

Thus,

$$u = \pm \frac{\partial \phi}{\partial x} \qquad v = \pm \frac{\partial \phi}{\partial y} \qquad w = \pm \frac{\partial \phi}{\partial z}$$

In cylindrical coordinates

$$V_r = \pm \frac{\partial \phi}{\partial r} \qquad V_\theta = \pm \frac{1}{r} \frac{\partial \phi}{\partial \theta} \qquad V_z = \pm \frac{\partial \phi}{\partial z}$$

The potential velocity, ϕ, exists only for irrotational flow. **Irrotationality may be a valid assumption for those regions of a flow in which viscous forces are negligible.** For example, such a region exists outside the boundary layer in the fluid over a solid surface.

All real fluids possess viscosity, but there are many situations in which the assumption of inviscid flow considerably simplifies the analysis and gives meaningful results.

STREAM FUNCTION AND VELOCITY POTENTIAL FOR TWO-DIMENSIONAL, IRROTATIONAL INCOMPRESSIBLE FLOW; LAPLACE'S EQUATION

For two dimensional, incompressible, inviscid flow, velocity components *u* and *v* can be expressed in terms of stream function, ψ, and the velocity potential, ϕ,

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u = \pm\frac{\partial \phi}{\partial x} \qquad v = \pm\frac{\partial \phi}{\partial y}$$

Substituting for *u* and *v* into the irrotational condition

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{we obtain}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{(A)}$$

Substituting for *u* and *v* into the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

we obtain

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{(B)}$$

Equations (A) and (B) are forms of **Laplace's equation**. Any function ψ or ϕ that satisfies Laplace's equation represents a possible two dimensional, incompressible, irrotational flow field.

Along a streamline, stream function ψ is constant, therefore

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

The slope of a streamline becomes

$$\left.\frac{dy}{dx}\right)_{\psi} = -\frac{\partial \psi / \partial x}{\partial \psi / \partial y} = -\frac{-v}{u} = \frac{v}{u}$$

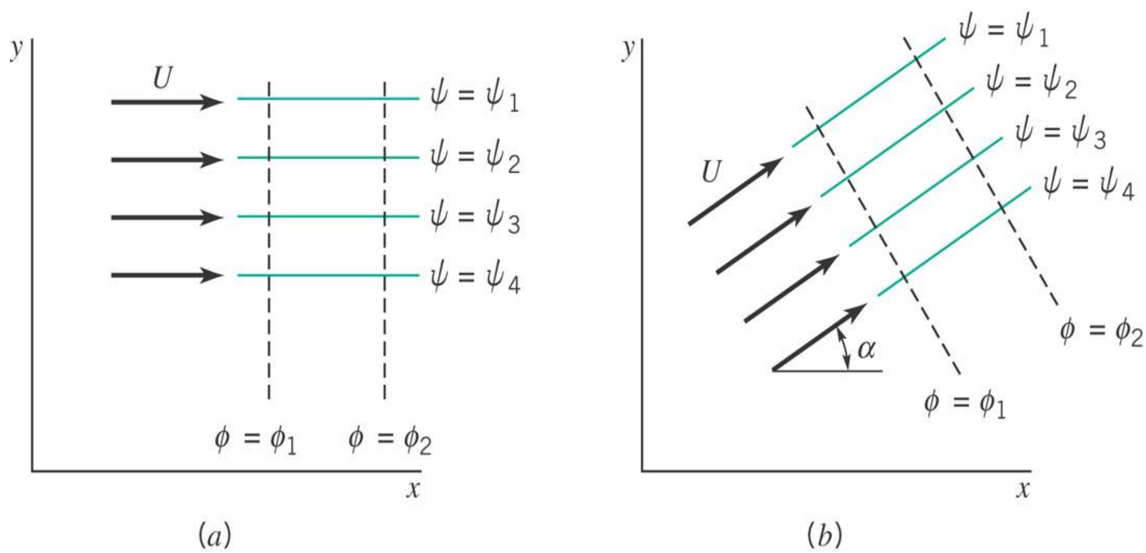
Along a line of constant ϕ, $d\phi = 0$ and

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

Consequently, the slope of a potential line becomes

$$\frac{dy}{dx} = -\frac{\partial \phi / \partial x}{\partial \phi / \partial y} = -\frac{u}{v}$$

As potential lines and streamlines have slopes that are negative reciprocals; they are perpendicular.

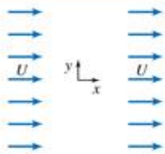


Example: Consider the flow field given by $\phi = 4x^2 - 4y^2$. Show that the flow **is irrotational**. Determine the stream function for this flow.

To be completed in class

ELEMENTARY PLANE FLOWS

A variety of potential flows can be constructed by superposing elementary flow patterns. The ψ and ϕ functions for five elementary two dimensional flows – a uniform flow, a source, a sink, a vortex and a doublet are summarized in the Table below.

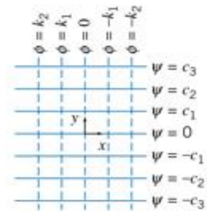


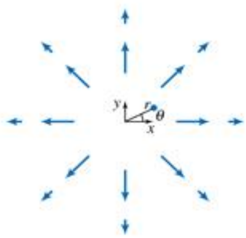
Uniform Flow (positive x direction)

$u = U \qquad \psi = Uy$

$v = 0 \qquad \phi = -Ux$

$\Gamma = 0$ around any closed curve





Source Flow (from origin)

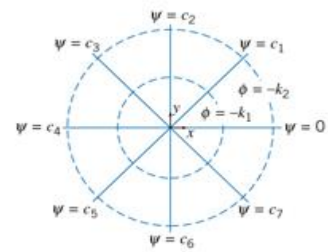
$V_r = \dfrac{q}{2\pi r} \qquad \psi = \dfrac{q}{2\pi}\theta$

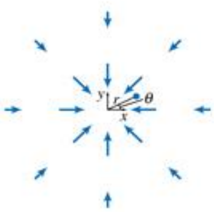
$V_\theta = 0 \qquad \phi = -\dfrac{q}{2\pi}\ln r$

Origin is singular point

q is volume flow rate per unit depth

$\Gamma = 0$ around any closed curve





Sink Flow (toward origin)

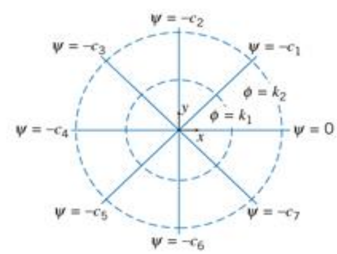
$V_r = -\dfrac{q}{2\pi r} \qquad \psi = -\dfrac{q}{2\pi}\theta$

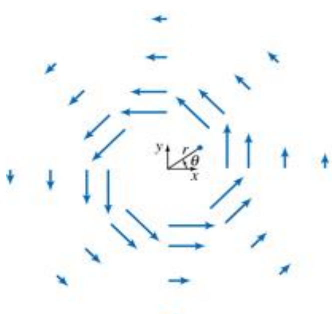
$V_\theta = 0 \qquad \phi = \dfrac{q}{2\pi}\ln r$

Origin is singular point

q is volume flow rate per unit depth

$\Gamma = 0$ around any closed curve





Irrotational Vortex (counterclockwise, center at origin)

$V_r = 0 \qquad \psi = -\dfrac{K}{2\pi}\ln r$

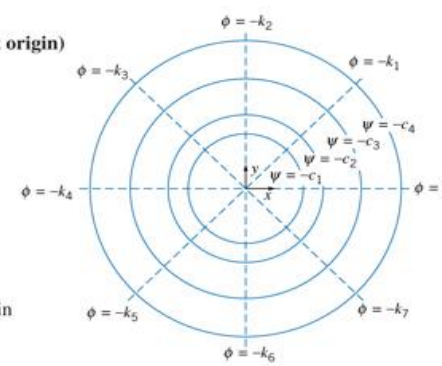
$V_\theta = \dfrac{K}{2\pi r} \qquad \phi = -\dfrac{K}{2\pi}\theta$

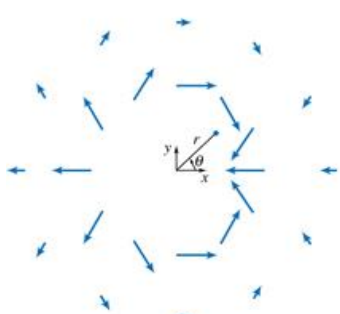
Origin is singular point

K is strength of the vortex

$\Gamma = K$ around any closed curve enclosing origin

$\Gamma = 0$ around any closed curve not enclosing origin





Doublet (center at origin)

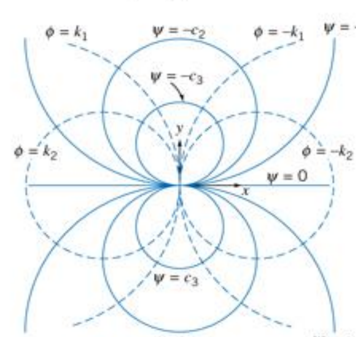
$V_r = -\dfrac{\Lambda}{r^2}\cos\theta \qquad \psi = -\dfrac{\Lambda\sin\theta}{r}$

$V_\theta = -\dfrac{\Lambda}{r^2}\sin\theta \qquad \phi = -\dfrac{\Lambda\cos\theta}{r}$

Origin is singular point

Λ is strength of the doublet

Γ = around any closed curve

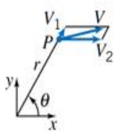


SUPERPOSITION OF ELEMENTARY PLANE FLOWS

We showed that both ϕ and ψ satisfy Laplace's equation for flow that is both incompressible and irrotational. Since Laplace's equation is a linear homogeneous partial differential equation, solutions may be superposed (added together) to develop more complex and interesting patterns of flows.

Table. Superposition of Elementary Plane Flows

Source and Uniform Flow (flow past a half-body)

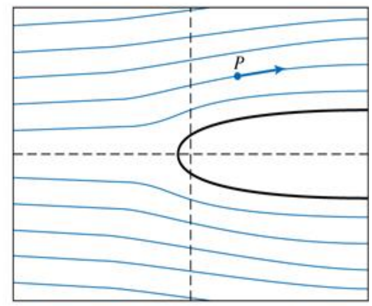


$$\psi = \psi_{so} + \psi_{uf} = \psi_1 + \psi_2 = \frac{q}{2\pi}\theta + Uy$$

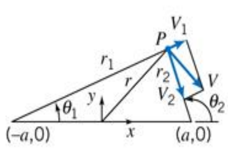
$$\psi = \frac{q}{2\pi}\theta + Ur\sin\theta$$

$$\phi = \phi_{so} + \phi_{uf} = \phi_1 + \phi_2 = -\frac{q}{2\pi}\ln r - Ux$$

$$\phi = -\frac{q}{2\pi}\ln r - Ur\cos\theta$$



Source and Sink (equal strength, separation distance on x axis = 2a)

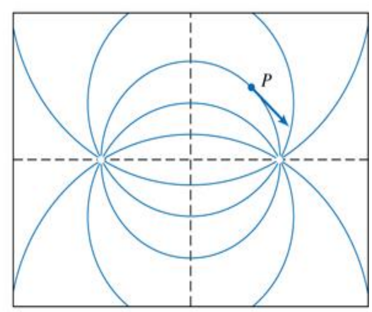


$$\psi = \psi_{so} + \psi_{si} = \psi_1 + \psi_2 = \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2$$

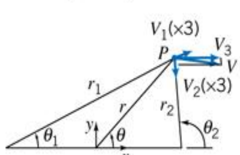
$$\psi = \frac{q}{2\pi}(\theta_1 - \theta_2)$$

$$\phi = \phi_{so} + \phi_{si} = \phi_1 + \phi_2 = -\frac{q}{2\pi}\ln r_1 + \frac{q}{2\pi}\ln r_2$$

$$\phi = \frac{q}{2\pi}\ln\frac{r_2}{r_1}$$



Source, Sink, and Uniform Flow (flow past a Rankine body)

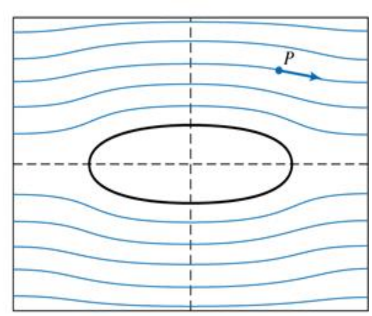


$$\psi = \psi_{so} + \psi_{si} + \psi_{uf} = \psi_1 + \psi_2 + \psi_3 = \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2 + Uy$$

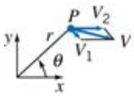
$$\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + Ur\sin\theta$$

$$\phi = \phi_{so} + \phi_{si} + \phi_{uf} = \phi_1 + \phi_2 + \phi_3 = -\frac{q}{2\pi}\ln r_1 + \frac{q}{2\pi}\ln r_2 - Ux$$

$$\phi = \frac{q}{2\pi}\ln\frac{r_2}{r_1} - Ur\cos\theta$$



Vortex (clockwise) and Uniform Flow

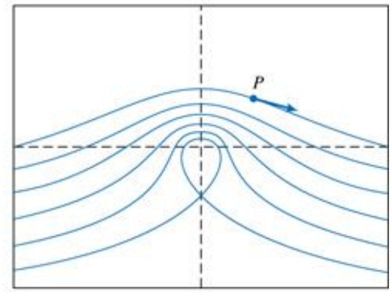


$$\psi = \psi_v + \psi_{uf} = \psi_1 + \psi_2 = \frac{K}{2\pi}\ln r + Uy$$

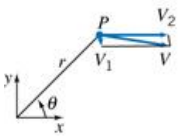
$$\psi = \frac{K}{2\pi}\ln r + Ur\sin\theta$$

$$\phi = \phi_v + \phi_{uf} = \phi_1 + \phi_2 = \frac{K}{2\pi}\theta - Ux$$

$$\phi = \frac{K}{2\pi}\theta - Ur\cos\theta$$



Doublet and Uniform Flow (flow past a cylinder)



$$\psi = \psi_d + \psi_{uf} = \psi_1 + \psi_2 = -\frac{\Lambda\sin\theta}{r} + Uy$$

$$= -\frac{\Lambda\sin\theta}{r} + Ur\sin\theta$$

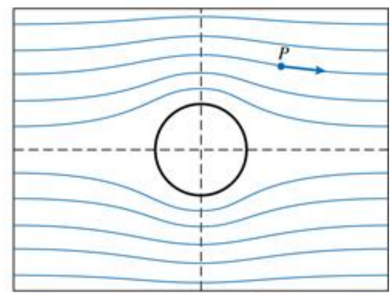
$$\psi = U\left(r - \frac{\Lambda}{Ur}\right)\sin\theta$$

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta \quad a = \sqrt{\frac{\Lambda}{U}}$$

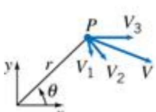
$$\phi = \phi_d + \phi_{uf} = \phi_1 + \phi_2 = -\frac{\Lambda\cos\theta}{r} - Ux$$

$$= -\frac{\Lambda\cos\theta}{r} - Ur\cos\theta$$

$$\phi = -U\left(r + \frac{\Lambda}{Ur}\right)\cos\theta = -Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta$$



Doublet, Vortex (clockwise), and Uniform Flow (flow past a cylinder with circulation)



$$\psi = \psi_d + \psi_v + \psi_{uf} = \psi_1 + \psi_2 + \psi_3$$

$$= -\frac{\Lambda\sin\theta}{r} + \frac{K}{2\pi}\ln r + Uy$$

$$\psi = -\frac{\Lambda\sin\theta}{r} + \frac{K}{2\pi}\ln r + Ur\sin\theta$$

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta + \frac{K}{2\pi}\ln r$$

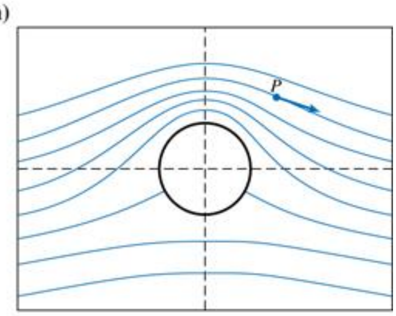
$$\phi = \phi_d + \phi_v + \phi_{uf} = \phi_1 + \phi_2 + \phi_3$$

$$= -\frac{\Lambda\cos\theta}{r} + \frac{K}{2\pi}\theta - Ux$$

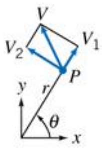
$a = \sqrt{\frac{\Lambda}{U}};\ K < 4\pi a U$

$$\phi = -\frac{\Lambda\cos\theta}{r} + \frac{K}{2\pi}\theta - Ur\cos\theta$$

$$\phi = -Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta + \frac{K}{2\pi}\theta$$

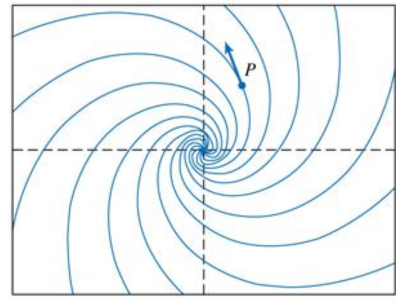


Source and Vortex (spiral vortex)

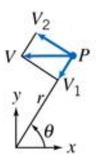


$$\psi = \psi_{so} + \psi_v = \psi_1 + \psi_2 = \frac{q}{2\pi}\theta - \frac{K}{2\pi}\ln r$$

$$\phi = \phi_{so} + \phi_v = \phi_1 + \phi_2 = -\frac{q}{2\pi}\ln r - \frac{K}{2\pi}\theta$$

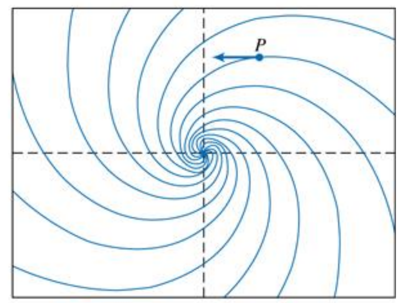


Sink and Vortex

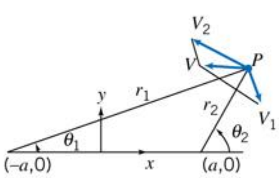


$$\psi = \psi_{si} + \psi_v = \psi_1 + \psi_2 = -\frac{q}{2\pi}\theta - \frac{K}{2\pi}\ln r$$

$$\phi = \phi_{si} + \phi_v = \phi_1 + \phi_2 = \frac{q}{2\pi}\ln r - \frac{K}{2\pi}\theta$$



Vortex Pair (equal strength, opposite rotation, separation distance on x axis = 2a)

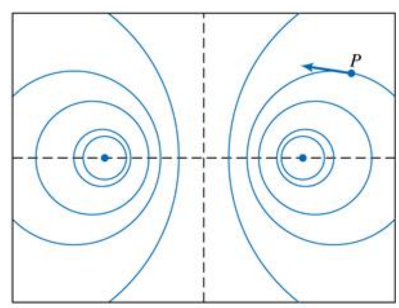


$$\psi = \psi_{v1} + \psi_{v2} = \psi_1 + \psi_2 = -\frac{K}{2\pi}\ln r_1 + \frac{K}{2\pi}\ln r_2$$

$$\psi = \frac{K}{2\pi}\ln\frac{r_2}{r_1}$$

$$\phi = \phi_{v1} + \phi_{v2} = \phi_1 + \phi_2 = -\frac{K}{2\pi}\theta_1 + \frac{K}{2\pi}\theta_2$$

$$\phi = \frac{K}{2\pi}(\theta_2 - \theta_1)$$



Example: A source with strength **0.2 m^3/s m** and a counterclockwise vortex with strength **1 m^3/s** are placed on origin. Obtain stream function an velocity potential, and velocity field for the combined flow. Find the velocity at point (1, 0.5).

To be completed in class

Example: The following stream function represents the flow past a cylinder of radius a with circulation.

$$\psi = UrSin\theta - \frac{Ua^2}{r} Sin\theta - aU \ln\left(\frac{r}{a}\right)$$

Determine the pressure distribution over the cylinder.

To be completed in class